# JOURNAL

OF THE

# INSTITUTE OF ACTUARIES

AND

# ASSURANCE MAGAZINE.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

## VOL. XVIII.

LONDON:
CHARLES & EDWIN LAYTON, 150 FLEET STREET.
NEW YORK: J. H. & C. M. GOODSELL, 156 BROADWAY; AND 156 & 157 WASHINGTON STREET, CHICAGO.
PARIS: ARMAND ANGER, LIBRAIRIE DES ASSURANCES, 48 RUE LAFFITTE.
HAMBURG: W. MAUKE SÖHNE. LA HAGUE: MARTINUS NIJHOFF.

1875.

*On the Arithmometer of M. Thomas (de Colmar), and its application to the Construction of Life Contingency Tables.* By PETER GRAY, F.R.A.S., F.R.M.S., *Honorary Member of the Institute of Actuaries.*

(Continued from page 32.)

PROBLEM VIII.—To construct a table for the formation of Survivorship Assurances.

A survivorship assurance on $(x)$ against $(y)$ is an assurance payable at the end of the year in which the combination $(x.y)$ is dissolved, provided the dissolution is caused by the death of $(x)$.

The value of this assurance, in respect of the $n$th year, is the value of the sum payable (supposed a unit) into the probability of the concurrence of these two events, namely, that $(x)$ shall die in that year, and that $(y)$ shall live over half of it. This gives

$$\frac{d_x\lambda_y v^n}{l_{x.y}},$$

in which $\lambda_y = \frac{1}{2}(l_y + l_{y+1})$.
Summation of this expression, for the proper values of $n$, will give the value of the assurance with reference to the whole or any portion of the future lifetime of $(x)$.

Making $n=1$, and multiplying numerator and denominator by $v^x$ for the compartment of the table in which $x$ is not less than $y$, and by $v^y$ for that in which $y$ is not less than $x$, we have the following expressions for the value in respect of the first year in the two cases:—

$$\frac{d_x\lambda_y v^{x+1}}{D_{x.y}}, \text{ and } \frac{d_x\lambda_y v^{y+1}}{D_{x.y}}.$$

Consequently, $D_{x.y}$ having been already formed, we have now to form only the numerators of these expressions, of the first for every combination of $x$ and $y$ in which $x$ is not less than $y$, and of the second for every combination in which $y$ is not less than $x$.

The following is the commencement of the two series $d_x$ and $\lambda_y$, with their differences. The former is given in the mortality table, and the latter is formed in accordance with the relation $\lambda_y = \frac{1}{2}(l_y + l_{y+1})$.

| $x$ & $y$ | $d_x$ | Δ | $\lambda_y$ | Δ |
|---|---|---|---|---|
| 97 | 9 | | 4·5 | |
| 6 | 40 | 31 | 29·0 | 24·5 |
| 5 | 86 | 46 | 92·0 | 63·0 |
| 4 | 139 | 53 | 204·5 | 112·5 |
| 3 | 195 | 56 | 371·5 | 167·0 |
| 2 | 254 | 59 | 596·0 | 224·5 |
| 1 | 329 | 75 | 887·5 | 291·5 |
| 0 | 408 | 79 | 1256·0 | 368·5 |
| 89 | 495 | 87 | 1707·5 | 451·5 |
| 8 | 615 | 120 | 2262·5 | 555·0 |
| 7 | 773 | 158 | 2956·5 | 694·0 |
| 6 | 941 | 168 | 3813·5 | 857·0 |
| 5 | 1138 | 197 | 4853·0 | 1039·5 |
| 4 | 1346 | 208 | 6095·0 | 1242·0 |
| 3 | 1545 | 199 | 7540·5 | 1445·5 |
| 2 | 1719 | 174 | 9172·5 | 1632·0 |
| 1 | 1883 | 164 | 10973·5 | 1801·0 |
| 0 | 2015 | 132 | 12922·5 | 1949·0 |
| * | * | * | * | * |

It may be well to remark that, the terms being arranged in reverse order, it is necessary, for the purpose of securing the correspondence of the differences with the ages to which they individually belong, to place them each against the term which constitutes the minuend in its formation. And they are all properly negative.

The formation for each age of the functions $d_x v^{x+1}$ and $\lambda_y v^{y+1}$ is the next step. For this purpose the Arithmometer comes into use. The commencement of the formation is here shewn, but it is so plain as not to stand in need of explanation.

| $x$ & $y$ | $v^{x+1}$ | $d_x$ | $d_x v^{x+1}$ | $\lambda_y$ | $\lambda_y v^{y+1}$ |
|---|---|---|---|---|---|
| 97 | 5520164 | 9 | 496815 | 45 | 24841 |
| 6 | 5685769 | 40 | 2274308 | 290 | 164887 |
| 5 | 5856342 | 86 | 5036454 | 920 | 538783 |
| 4 | 6032032 | 139 | 8384524 | 2045 | 1233551 |
| 3 | 6212993 | 195 | 12115336 | 3715 | 2308127 |
| 2 | 6399383 | 254 | 16254433 | 5960 | 3814032 |
| 1 | 6591364 | 329 | 21685588 | 8875 | 5849836 |
| 0 | 6789105 | 408 | 27699548 | 12560 | 8527116 |
| 89 | 6992779 | 495 | 34614256 | 17075 | 11940170 |
| 8 | 7202562 | 615 | 44295756 | 22625 | 16295797 |
| 7 | 7418639 | 773 | 57346079 | 29565 | 21933206 |
| 6 | 7641198 | 941 | 71903673 | 38135 | 29139709 |
| 5 | 7870434 | 1138 | 89565539 | 48530 | 38195216 |
| 4 | 8106547 | 1346 | 91141226 | 60950 | 49409404 |
| 3 | 8349743 | 1545 | 129003529 | 75405 | 62961237 |
| 2 | 8600236 | 1719 | 147838057 | 91725 | 78885665 |
| 1 | 8858243 | 1883 | 166800716 | 109735 | 97205930 |
| 0 | 9123990 | 2015 | 183848399 | 129225 | 117904761 |
| * | * | * | * | * | * |

We are now prepared for the principal formations.

When $x$ is equal to or greater than $y$, we have, if $y$ vary,

$$\Delta_y d_x \lambda_y v^{x+1} = d_x \lambda_y v^{x+1} . \Delta\lambda_y ;$$

whence

$$d_x \lambda_{y+1} v^{x+1} = d_x \lambda_y v^{x+1} + d_x \lambda_y v^{x+1} . \Delta\lambda_y ;$$

and for the formation in reverse order, by transposition,

$$d_x \lambda_y v^{x+1} = d_x \lambda_{y+1} v^{x+1} - d_x \lambda_y v^{x+1} . \Delta\lambda_y .$$

Finally, changing $y$ into $y-1$,

$$d_x \lambda_{y-1} v^{x+1} = d_x \lambda_y v^{x+1} - d_x \lambda_{y-1} v^{x+1} . \Delta\lambda_{y-1} .$$

By this formula, which is of the form P − QR, the required values are to be constructed; but $\Delta\lambda_{y-1}$ being negative, the regulator has to be set for addition.

The following is a specimen of the process.

$x > y$

| $x$ | $d_x v^{x+1}$ | $\lambda_y$ | $x$ | Diff. 0. | Diff. 1. | Diff. 2. | Diff. 3. | Diff. 4. |
|---|---|---|---|---|---|---|---|---|
| 97 | 49682 | 45 | 97 | ·02236 | ·14408 | ·45707 | 1·01600 | 1·84569 |
| 6 | 227431 | 245<br>290 | | ·65955 | 2·09236 | 4·65094 | 8·44902 | 13·55483 |
| 5 | 503645 | 630<br>920 | 6 | ·68191<br>4·63354 | 2·23644<br>10·29956 | 5·10801<br>18·71044 | 9·46502<br>30·01730 | 15·40052<br>44·69858 |
| 4 | 838452 | 1125<br>2045 | 5 | 5·31545<br>17·14634 | 12·53600<br>31·14849 | 23·81845<br>49·9717 | 39·48232<br>74·4126 | 60·09910<br>105·3096 |
| 3 | 1211534 | 1670<br>3715 | 4 | 22·46179<br>45·0085 | 43·68449<br>72·2074 | 73·7902<br>107·5236 | 113·8949<br>152·1687 | 165·4087<br>206·8694 |
| 2 | 1625443 | 2245<br>5960 | 3 | 67·4703<br>96·8765 | 115·8919<br>144·2582 | 181·3138<br>204·1558 | 266·0636<br>277·5446 | 372·2781<br>367·7567 |
| 1 | 2168559 | 2915<br>8875 | 2 | 164·3468<br>192·4595 | 260·1501<br>272·3709 | 385·4696<br>370·2813 | 543·6082<br>490·6362 | 740·0348<br>641·1342 |
| 0 | 2769955 | 3685<br>12560 | 1 | 356·8063<br>347·9062 | 532·5210<br>472·9696 | 755·7509<br>626·7021 | 1034·2444<br>818·9369 | 1381·1690<br>1056·322 |
| 89 | 3461426 | 4515<br>17075 | 0 | 704·7125<br>591·0385 | 1005·4906<br>783·1476 | 1382·4530<br>1023·3706 | 1853·1813<br>1320·0148 | 2437·491<br>1679·830 |
| 8 | 4429576 | 5550<br>22625 | 89 | 1295·7510<br>1002·192 | 1788·6382<br>1309·604 | 2405·8236<br>1689·219 | 3173·1961<br>2149·673 | 4117·321<br>2699·827 |
| 7 | 5734608 | 6940<br>29565 | 8 | 2297·943<br>1695·437 | 3098·242<br>2186·893 | 4095·043<br>2783·005 | 5322·869<br>3495·244 | 6817·148<br>4324·181 |
| 6 | 7190367 | 8570<br>38135 | 7 | 3993·380<br>2742·047 | 5285·135<br>3489·486 | 6878·048<br>4382·529 | 8818·113<br>5421·897 | 11141·329<br>6595·365 |
| 5 | 8956554 | 10395<br>48530 | 6 | 6735·427<br>4346·616 | 8774·621<br>5459·020 | 11260·577<br>6753·690 | 1424·0010<br>821·5399 | 17736·694<br>9828·475 |
| | | | 5 | 11082·043 | 14233·641 | 18014·267 | 22455·409 | 27565·169 |
| * | * | * | * | * | * | * | * | * |

Here, as in the example of the last problem, to facilitate the subsequent summations, $x$ is constant in the rows, and $x-y$ in the columns. The results, as they arise, have consequently to be written in the rows, commencing in each case in the first column, Diff. 0. There are in the example two columns headed $x$. The ages in the first of these columns are opposite the rows in the principal columns occupied by the successive terms of $d_x\lambda_y v^{x+1}$, and the ages in the second are opposite the rows occupied by the summations of the same function. The columns between those containing the values of $x$, are occupied, as shown, by the terms of two of the series whose formation has been exemplified.

The process of formation is as follows. To form the row corresponding to any given value of $x$, take the proper term of $d_x v^{x+1}$ for the in-factor, and multiply continuously by the adjoining term of $\lambda_y$, and the differences of this series, commencing with $\Delta\lambda_{y-1}$. The regulator is at addition throughout, since the differences of $\lambda_y$ are negative. A slip containing the differences should be placed on the machine in front of the operator; and to avoid error, each difference as it is used should be covered with a paper weight.*

Series for verification may be formed as follows :—

The function is

$$d_x\lambda_y v^{x+1}.$$

Hence if $x$ vary,

$$\Delta_x(d_x\lambda_y v^{x+1}) = \lambda_y . \Delta(d_x v^{x+1});$$

$$\therefore \quad d_{x+1}\lambda_y v^{x+2} = d_x\lambda_y v^{x+1} + \lambda_y\Delta(d_x v^{x+1};$$

or, changing $x$ into $x-1$,

$$d_x\lambda_y v^{x+1} = d_{x-1}\lambda_y v^x + \lambda_y\Delta(d_{x-1}v^x).$$

The series thus indicated is one that, commencing in Column 0, $(x=y,)$ ascends diagonally. The formation, commencing with age 85, is as follows :—

* A better plan, perhaps, will be to insert the differences, as in the example, between the terms from which they arise.

Here, as in the example of the last problem, to facilitate the subsequent summations, $x$ is constant in the rows, and $x-y$ in the columns. The results, as they arise, have consequently to be written in the rows, commencing in each case in the first column, Diff. 0. There are in the example two columns headed $x$. The ages in the first of these columns are opposite the rows in the principal columns occupied by the successive terms of $d_x\lambda_y v^{x+1}$, and the ages in the second are opposite the rows occupied by the summations of the same function. The columns between those containing the values of $x$, are occupied, as shown, by the terms of two of the series whose formation has been exemplified.

The process of formation is as follows. To form the row corresponding to any given value of $x$, take the proper term of $d_x v^{x+1}$ for the in-factor, and multiply continuously by the adjoining term of $\lambda_y$, and the differences of this series, commencing with $\Delta\lambda_{y-1}$. The regulator is at addition throughout, since the differences of $\lambda_y$ are negative. A slip containing the differences should be placed on the machine in front of the operator; and to aveid error, each difference as it is used should be covered with a paper weight.*

Series for verification may be formed as follows :—

The function is

$$d_x\lambda_y v^{x+1}.$$

Hence if $x$ vary,

$$\Delta_x(d_x\lambda_y v^{x+1})=\lambda_y\,.\,\Delta(d_x v^{x+1})\,;$$

$$\therefore\quad d_{x+1}\lambda_y v^{x+2}=d_x\lambda_y v^{x+1}+\lambda_y\Delta(d_x v^{x+1}\,;$$

or, changing $x$ into $x-1$,

$$d_x\lambda_y v^{x+1}=d_{x-1}\lambda_y v^x+\lambda_y\Delta(d_{x-1}v^x).$$

The series thus indicated is one that, commencing in Column 0, ($x=y$,) ascends diagonally. The formation, commencing with age 85, is as follows :—

* A better plan, perhaps, will be to insert the differences, as in the example, between the terms from which they arise.

| $x$ | $\Delta d_x v^{x+1}$ | $\lambda_{85} d_x v^{x+1}$ |
|---|---|---|
| 85 | | 4346616 |
| 6 | 1766187 | 3489486 |
| 7 | 1455759 | 2783005 |
| 8 | 1305032 | 2149673 |
| 9 | 968150 | 1679830 |
| 90 | 691471 | 1344259 |
| 1 | 601396 | 1052401 |
| 2 | 543116 | 7888280 |
| 3 | 413909 | 5879575 |
| 4 | 373082 | 4069008 |
| 5 | 334807 | 2444194 |
| 6 | 276214 | 1103718 |
| 7 | 177749 | 241107 |
| | 49682 | 000000 |

Taking $\lambda_{85}$ for the in-factor, we have here for the initial term $\lambda_{85} \times d_{85}v^{86}$, $=4346616$; and the succeeding terms are formed by continuous multiplication by the differences of the series $d_x v^{x+1}$, the regulator being set for subtraction. Verification is had by the term beyond 97 coming out equal to 0.

We have now to attend to the construction of the values of the function for the combinations in which $x$ is equal to or less than $y$.

The function is,

$$d_x \lambda_y v^{y+1};$$

and if $x$ vary we have,

$$\Delta_x (d_x \lambda_y v^{y+1}) = \lambda_y v^{y+1}.\Delta d_x.$$

Whence,

$$d_{x+1}\lambda_y v^{y+1} = d_x \lambda_y v^{y+1} + \lambda_y v^{y+1}.\Delta d_x;$$

and for the formation in reverse order,

$$d_x \lambda_y v^{y+1} = d_{x+1}\lambda_y v^{y+1} - \lambda_y v^{y+1}.\Delta d_x.$$

And changing $x$ into $x-1$,

$$d_{x-1}\lambda_y v^{y+1} = d_x \lambda_y v^{y+1} - \lambda_y v^{y+1}.\Delta d_{x-1},$$

which is the working formula.

The following is an example, showing the commencement of the process.

$x<y$

| $y$ | $\lambda_y v^{y+1}$ | $d_x$ | $y$ | Diff. 0. | Diff. 1. | Diff. 2. | Diff. 3. | Diff. 4. |
|---|---|---|---|---|---|---|---|---|
| 97 | 24841 | 9 | 97 | ·02236 | ·09936 | ·21363 | ·34529 | ·48440 |
| | | 31 | | | | | | |
| 6 | 164887 | 40 | | ·65955 | 1·41803 | 2·29194 | 3·21531 | 4·18814 |
| | | 46 | 6 | ·68191 | 1·51739 | 2·50557 | 3·56060 | 4·67254 |
| 5 | 538783 | 86 | | 4·63354 | 7·48909 | 10·50628 | 13·68510 | 17·72598 |
| | | 53 | 5 | 5·31545 | 9·00648 | 13·01185 | 17·24570 | 22·39852 |
| 4 | 1233551 | 139 | | 17·14635 | 24·05423 | 31·33218 | 40·58381 | 50·32886 |
| | | 56 | 4 | 22·46180 | 33·06071 | 44·34403 | 57·82951 | 72·72738 |
| 3 | 2308127 | 195 | | 45·00848 | 58·62643 | 75·93738 | 94·17158 | 114·2523 |
| | | 59 | 3 | 67·47028 | 91·68714 | 120·28141 | 152·00109 | 186·9797 |
| 2 | 3814032 | 254 | | 96·87643 | 125·4817 | 155·6125 | 188·7946 | 234·5630 |
| | | 75 | 2 | 164·34671 | 217·1688 | 275·8939 | 340·7957 | 421·5427 |
| 1 | 5849836 | 329 | | 192·4596 | 238·6733 | 289·5668 | 359·7649 | 452·1922 |
| | | 79 | 1 | 356·8063 | 455·8421 | 565·4607 | 700·5606 | 873·7349 |
| 0 | 8527116 | 408 | | 347·9063 | 422·0922 | 524·4176 | 659·1460 | 802·4015 |
| | | 87 | 0 | 704·7126 | 877·9343 | 1089·8783 | 1359·7066 | 1676·1364 |
| 89 | 1194017 | 495 | | 591·0384 | 734·3205 | 922·9751 | 1123·570 | 1358·791 |
| | | 120 | 89 | 1295·7510 | 1612·2548 | 2012·8534 | 2483·277 | 3034·927 |
| 8 | 1629580 | 615 | | 1002·191 | 1259·665 | 1533·434 | 1854·461 | 2193·414 |
| | | 158 | 8 | 2297·942 | 2871·920 | 3546·287 | 4337·738 | 5228·341 |
| 7 | 2193321 | 773 | | 1695·437 | 2063·915 | 2495·999 | 2952·209 | 3388·680 |
| | | 168 | 7 | 3993·379 | 4935·835 | 6042·286 | 7289·947 | 8617·021 |
| 6 | 2913971 | 941 | | 2742·047 | 3316·099 | 3922·204 | 4502·085 | 5009·116 |
| | | 197 | 6 | 6735·426 | 8251·934 | 9964·490 | 11792·032 | 13626·137 |
| 5 | 3819522 | 1138 | | 4346·615 | 5141·076 | 5901·161 | 6565·757 | 7192·159 |
| | | | 5 | 11082·041 | 13393·010 | 15865·651 | 18357·789 | 20818·296 |
| * | * | * | * | * | * | * | * | * |

The formation here is in such entire accordance with that of $d_x\lambda_y v^{x+1}$, already given, that explanation of it would seem to be superfluous. Comparison of the two examples will show that they differ only by the interchange of $x$ and $y$, $d_x v^{x+1}$ and $\lambda_y v^{y+1}$, and $\lambda_y$ and $d_x$. The process in both is the same. The leading term of each row is the product of the values of the two functions in line with it; and the succeeding terms are produced by the continuous employment as multipliers of the series of differences in the column adjoining that headed Diff. 0.

The formation of the verification series here is also in accordance with that of the series for the verification of $d_x\lambda_y v^{x+1}$.

| $y$ | $\Delta\lambda_y v^{y+1}$ | $d_{85}\lambda_y v^{y+1}$ |
|---|---|---|
| 85 | | 4346615 |
| 6 | 905551 | 3316099 |
| 7 | 720650 | 2495999 |
| 8 | 563741 | 1854461 |
| 9 | 435563 | 1358791 |
| 90 | 341305 | 9703857 |
| 1 | 267728 | 6657112 |
| 2 | 203581 | 4340369 |
| 3 | 150590 | 2626649 |
| 4 | 107458 | 1403780 |
| 5 | 69477 | 613136 |
| 6 | 37389 | 187642 |
| 7 | 14005 | 28269 |
| | 2484 | 00000 |

The formation of the functions $d_x\lambda_y v^{x+1}$ and $d_x\lambda_y v^{y+1}$ having been completed, there remain the summations of the terms to be performed. This is rendered a very simple matter in consequence of the arrangement adopted, which is that suggested by General Hannyngton in the case of the construction of $\mathrm{D}_{x \cdot y}$. To ensure accuracy the columns should first be added in groups of ten or fifteen terms; and the sums being carried forward, will serve as checks in the final summation.

It is the results of these final summations that are tabulated, each of them having reference to the entire after-lifetime of the combination opposite which it is placed.

To suit the form of the table, it is requisite now to make a change in the nomenclature that up to this point has been employed. The symbols $x$ and $y$ have been used hitherto to denote the ages of the life assured and the life assured against, respectively. This appropriation of the symbols was convenient so long as we were concerned with merely the construction of the terms; but it is otherwise when we come to consider the arrangement which will most facilitate the use of the table. The only tables of this kind that have been published are the valuable and extensive tables of Mr. David Chisholm, founded on the Carlisle Table of Mortality; and we cannot do better than adopt the arrangement and notation of which Mr. Chisholm has set the example in his work. In the sequel, therefore, when the ages are other than equal, $x$ will stand for the older of the two, and $y$ for the younger; and the two sets of results will be arranged in parallel columns, designated respectively by the symbols $\mathrm{M}^{1}_{\overline{x.y}}$ and $\mathrm{M}_{\overline{x.y}}^{\;\;1}$, the analogy of which to those which denote the values of Survivorship

Assurances on $(x)$ and $(y)$ is apparent. We may also, if we please, form a column $M_{x \cdot y}$, by adding together the corresponding terms in the two preceding columns.

The following is a specimen of the final arrangement.

Diff. 2.

| $x$ | $y$ | $M_{\overline{x.y}}^{1}$ | $M_{x.\overline{y}}^{\;\;1}$ | $M_{x.y}$ |
|---|---|---|---|---|
| * | * | * | * | * |
| 85 | 83 | 18014·27 | 15865·65 | 33879·92 |
| 86 | 84 | 11260·58 | 9964·49 | 21225·07 |
| 87 | 85 | 6878·05 | 6042·29 | 12920·34 |
| 88 | 86 | 4095·04 | 3546·29 | 7641·33 |
| 89 | 87 | 2405·82 | 2012·85 | 4418·67 |
| 90 | 88 | 1382·45 | 1089·88 | 2472·33 |
| 91 | 89 | 755·751 | 565·461 | 1321·212 |
| 92 | 90 | 385·470 | 275·894 | 661·364 |
| 93 | 91 | 181·314 | 120·281 | 301·595 |
| 94 | 92 | 73·790 | 44·344 | 118·134 |
| 95 | 93 | 23·818 | 13·012 | 36·830 |
| 96 | 94 | 5·108 | 2·506 | 7·614 |
| 97 | 95 | ·457 | ·214 | ·671 |

In vol. v of the *Journal*, pp. 107 to 118, will be found a prototype of the operation that forms the subject of the present problem.

It is in place to mention that the examples here given, like those in illustration of Problem VII. are pointed to correspond to a radix of 10,000.

I have now completed the task I proposed to myself when commencing the preparation of the present series of papers. I have, I think, shown that the Arithmometer possesses a singular adaptation to the construction of the tables required for Actuarial use. The examples I have given by no means exhaust the capabilities of the instrument in this department; but the necessity for here going further into the subject is precluded by the publication, since my papers were commenced, of Mr. Ralph P. Hardy's remarkable work, entitled *Valuation Tables, based upon the Institute of Actuaries' Mortality Experience* ($H^M$) *Table.* Mr. Hardy's work contains complete tables, at four rates of interest, of most of the functions which I have chosen for examples, and of others besides, nearly all of them having been calculated by the aid of the Arithmometer. It thus forms an extensive repertory of examples for the use of such as desire to master the working of the machine, and to elicit its further capabilities. Had the work referred to been published, or had I known that it was in contemplation, before I commenced the preparation of my papers, they would, if they had appeared at all, have assumed a different form from that which they now present. Mr. Hardy's work is one of unquestionable utility, and will form a lasting monument of the author's skill and enterprise.

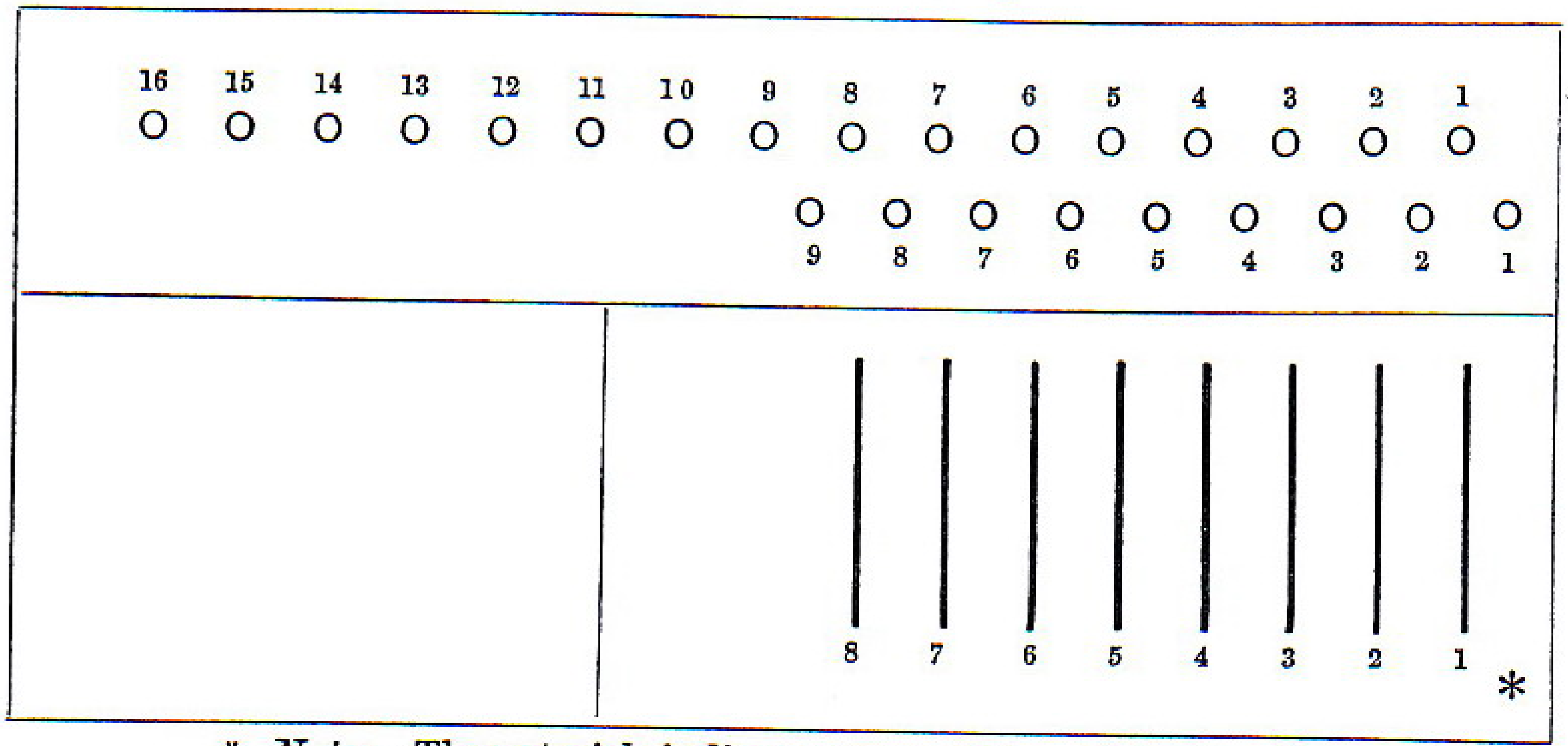
It will be sufficiently apparent, from what I have said, that I quite concur in the laudatory terms in which Mr. Hardy, in his preface, speaks of the Arithmometer. I also agree with him in what he says as to the existence of room for improvement in the strength and temper of certain of the materials employed. This refers chiefly to the springs, which form the weak point of the machine. Any improvement in form or material given to these, which should remove or diminish the tendency to give way that they sometimes exhibit, would be hailed as a boon by all users of the machine.

I hope it will not be deemed presumptuous if now, ere I close, I venture respectfully to invite the attention of the makers of the machine to two points, in regard to which I consider some modification is desirable.

The first has reference to the manner in which the result (when the process in use is multiplication) is presented. The figures composing it are seen at the bottom of a series of holes, that which I have designated $S_1$, being the upper series on the slide. These holes are of the form of truncated hemispherical cavities, a quarter of an inch deep (such being the thickness of the slide), and five-eighths of an inch in diameter; and a consequence of this form is, that in working, with the paper at the right hand, and especially if the numbers being dealt with are *long* ones, more movement of the body, and more stooping over the machine than is at all agreeable, become necessary, to avoid mistake in reading off the figures. A position and direction of the light, also, which, probably, cannot always be conveniently commanded, are required for a like reason.

The remedy for these inconveniences—for such I believe they will be admitted to be by all who have had occasion to use the machine extensively—would be, of course, a reduction in the depth of the cavities at the bottom of which the figures appear. I do not presume to suggest the manner in which this might be accomplished; I will only say that I see no insurmountable mechanical difficulty in the way.

The second point to which I refer, is the desirableness of having the openings through which the figures appear individually numbered. To render my meaning plain, I give here a plan of the machine, with the figures that I proposed inserted.




* *Note.*—The asterisk indicates the position of the handle.

I frequently feel the want of this provision in setting factors on the machine, having, in its absence, to count the holes; and also in performing multiplication from left to right.* I believe that those practically acquainted with the machine will agree with me in thinking that the additions I have suggested would facilitate its use.

* I find this the more convenient order, and I always practise it when it is *safe* to do so—that is, when the number on $S_1$ does not extend beyond the limit of the carrying power of the machine; and also even when it does so extend, provided that the leading figures change but slowly.

---

## *On the Solution of Problems connected with Loans repayable by Instalments.* *By* W. M. MAKEHAM, F.I.A.

THE only treatise of any value on this subject that I am acquainted with is a paper by Mr. Peter Gray, in vol. xiv of the *Journal of the Institute of Actuaries*. Considering the magnitude of the pecuniary interests frequently involved in transactions of the nature referred to, it is somewhat surprising that Mr. Gray's valuable labours have not, ere this, been followed up by contributions from other writers on the doctrine of Interest and Annuities. This, I think, is much to be regretted,—and must form my apology for the following attempt to add something to what Mr. Gray has given us.

---

To find the value of a loan repayable by instalments at stated periods of time, with interest in the meantime at the rate $i$ on the